On formation of focus wave modes

1994 J. Phys. A: Math. Gen. 27 2587

(http://iopscience.iop.org/0305-4470/27/7/033)

J. Phys. A: Math. Gen. 27 (1994) 2587–2591. Printed in the UK

On formation of focus wave modes

Victor V Borisov† and Andrei B Utkin‡§

† Institute of Physics, St Petersbug University, Ulyanovskaya 1, Petrodvorets, St Petersburg, 198904, Russia
‡ Institute for Laser Physics, Scientific Center *S.I. Vavilov State Optics Institute*, Birzhevaya 12, St Petersburg, 199034, Russia

Received 2 September 1993

Abstract. The solution of the inhomogeneous wave equation is found. The source term is a pulse which has the velocity of light. The solution is a function which describes both transient and steady-state wave processes. The latter corresponds to Brittingham's axisymmetric focus wave mode.

1. Introduction

Brittingham [1] has found solutions of source-free Maxwell equations which describe electromagnetic waves localized in free space (so-called Brittingham's focus wave modes). Sezginer [2] has derived expressions for focus wave modes, which are solutions of the above equations, satisfying some special conditions. He has obtained a solution of the homogeneous wave equation where the wavefunction is taken as the Hertz potential. Hillion [3–5] has discussed various solutions of the homogeneous wave and Maxwell's equations with respect to the focus wave mode problem.

The purpose of this work is the following:

(i) To derive an expression for the axisymmetric Brittingham's focus wave mode as a solution of the inhomogeneous equation. A source of some special form, being turned on at an initial instant of time, provides the necessary wave.

(ii) To obtain relations which describe both transient and steady-state waves. The latter corresponds to Brittingham's focus wave mode of order zero. Therefore, this mode can be treated as a steady-state wave with a complicated time dependence.

(iii) To define the domain where the above Brittingham's focus wave mode exists.

We derive a solution to the scalar wave equation. Vectors of the electromagnetic field can be obtained from the scalar wavefunction.

2. General expression for the wavefunction

For the axisymmetric problem the wave equation for the function ψ in cylindric coordinates ρ, φ, z is

$$\left(\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial z^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\right)\psi=\frac{4\pi}{c}f \tag{1}$$

§ E-mail: utkin@soi.spb.su (Internet).

where $\tau = ct$ is the time variable (c is the velocity of light). Let f be a pulse with Gaussian transverse variation,

$$f = \frac{1}{2\pi}\,\delta(\tau - z)J(z,\tau)\exp(-\rho^2/\alpha^2)h(z) \qquad \tau > 0$$
$$f \equiv 0 \qquad \tau < 0. \tag{2}$$

Here $\delta(x)$ is the Dirac distribution, a is a parameter, $h(x)$ is the Heaviside step function, and $J(z, \tau)$ is a continuous function. The initial condition is

$$\psi = 0 \qquad \tau < 0. \tag{3}$$

Using the Fourier–Bessel transform

$$\Phi(s, z, \tau) = \int_0^\infty \mathrm{d}\rho\ \rho J_0(s\rho)\Phi(p, z, \tau)$$

$$\Phi(\rho, z, \tau) = \int_0^\infty \mathrm{d}s\ sJ_0(s\rho)\Phi(s, z, \tau)$$

where $J_0(s\rho)$ is the Bessel function of the first kind of order zero, Φ is either ψ or f, and the relation (see item 6.631 of Grandshteyn and Ryzhik [6])

$$\int_0^\infty \mathrm{d}x\ x\exp(-\alpha x^2)J_0(\beta x) = \frac{1}{2\alpha}\exp\left(-\frac{\beta^2}{4\alpha}\right)$$

one obtains from (1), (2) and (3)

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial z^2} + s^2\right)\psi(s, z, \tau) = \frac{\alpha^2}{c}\,\delta(\tau - z)J(z,\tau)\exp\left(-\frac{\alpha^2 s^2}{4}\right)h(z)$$

with the initial condition

$$\psi(s, z, \tau) \equiv 0 \qquad \tau < 0.$$

The solution of the latter problem can be obtained with the help of the Riemann method [7] (the Riemann function is $J_0(s\sqrt{(\tau-\tau')^2-(z-z')^2})$)

$$\psi(s, z, \tau) = \frac{\alpha^2}{2c}\exp\left(-\frac{\alpha^2 s^2}{4}\right)\int_0^\tau \mathrm{d}\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z'\ \delta(\tau'-z')h(z')J(z',\tau')$$
$$\times J_0(s\sqrt{(\tau-\tau')^2-(z-z')^2}).$$

So, coming back to the ρ-representation, one has

$$\psi(\rho, z, \tau) = \frac{\alpha^2}{2c}\int_0^\infty \mathrm{d}s\ s\exp\left(-\frac{\alpha^2 s^2}{4}\right)J_0(s\rho)$$
$$\times\int_0^\tau \mathrm{d}\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z'\ \delta(\tau'-z')h(z')J(z',\tau')J_0(s\sqrt{(\tau-\tau')^2-(z-z')^2}).$$

Rearranging the integrals,

$$\psi(\rho, z, \tau)=\frac{\alpha^2}{2c}\int_0^{\tau} d\tau' \int_{\tau'+2-\tau}^{-\tau'+z+\tau} dz'\, \delta(\tau'-z')h(z')J(z', \tau')$$
$$\times \int_0^{\infty} ds\, s \exp\left(-\frac{\alpha^2 s^2}{4}\right) J_0(s\rho) J_0[s\sqrt{(\tau-\tau')^2-(z-z')^2}]$$

and using the relation (see result 6.633 in [6])

$$\int_0^{\infty} dx\, x \exp(-\gamma^2 x^2) J_0(\alpha x) J_0(\beta x)=\frac{1}{2\gamma^2}\exp\left(-\frac{\alpha^2+\beta^2}{4\gamma^2}\right) J_0\left(i\,\frac{\alpha\beta}{2\gamma^2}\right)$$

we have

$$\psi(\rho, z, \tau)=\frac{1}{c}\exp\left(-\frac{\rho^2}{\alpha^2}\right)\int_0^{\tau} d\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} dz'\, \delta(\tau'-z')h(z')J(z', \tau')$$
$$\times \exp\left(-\frac{1}{\alpha^2}[(\tau-\tau')^2-(z-z')^2]\right) J_0\left[i\,\frac{2\rho}{\alpha^2}\sqrt{(\tau-\tau')^2-(z-z')^2}\right]. \tag{4}$$

Using the characteristic coordinates $\xi_{1,2}=\tau \mp z$ and $\xi'_{1,2}=\tau' \mp z'$, one can obtain for the above solution of the problem (1), (3)

$$\psi(\rho, z, \tau)=\frac{1}{2c}\exp\left(-\frac{1}{\alpha^2}(\rho^2+\xi_1\xi_2)\right)\int_0^{\xi_2} d\xi'_2\, J(0, \xi'_2)$$
$$\times \exp\left(\frac{1}{\alpha^2}\xi_1\xi'_2\right) J_0\left(i\,\frac{2\rho}{\alpha^2}\sqrt{\xi_1(\xi_2-\xi'_2)}\right) \tag{5}$$

where the integrand factor J, describing the longitudinal source distribution, is taken as a function of $\xi_{1,2}$ and, then, ξ_1 is replaced by zero while ξ_2 is replaced by ξ'_2.

3. The focus wave mode of order zero

Let $J(0, \xi'_2)=\exp(ik\xi'_2)$, where $k=\omega/c$ is a parameter. Substituting the above expression in (5) one obtains

$$\psi(\rho, \xi_1, \xi_2)=\frac{1}{2c}\exp\left(-\frac{\rho^2+\xi_1\xi_2}{\alpha^2}\right)$$
$$\times \int_0^{\xi_2} d\xi'_2 \exp\left[i\left(k-i\,\frac{\xi_1}{\alpha^2}\right)\xi'_2\right] J_0\left[i\,\frac{2\rho}{\alpha^2}\sqrt{\xi_1(\xi_2-\xi'_2)}\right].$$

Using the relation (see item 16.53 in [8])

$$U_n(w, \mu) \pm i U_{n+1}(w, \mu)=\frac{w^n}{\mu^{n-1}}\int_0^1 dt\, t^n J_{n-1}(\mu t)\exp\left(\pm\frac{1}{2} i w(1-t^2)\right)$$

where $U_n(w, \mu)$ is Lommel's function of two variables of order n we have

$$\psi(\rho, \xi_1, \xi_2)=\frac{1}{c}\exp\left(-\frac{\rho^2+\xi_1\xi_1}{\alpha^2}\right)\frac{1}{2(k-\mathrm{i}\xi_1/\alpha^2)}[U_1(w, \mu)+\mathrm{i}U_2(w, \mu)] \tag{6}$$

with $w=2(k-\mathrm{i}\xi_1/\alpha^2)\xi_2$, $\mu=\mathrm{i}(2\rho/\alpha^2)\sqrt{\xi_1\xi_2}$. The last expression is the scalar wavefunction which describes both transient and steady-state processes.

For $|w/\mu|<1$ the function $U_n(w, \mu)$ can be represented in the form

$$U_n(w, \mu)=\sum_{m=0}^{\infty}(-1)^m(w/\mu)^{n+2m}J_{n+2m}(\mu)$$

while for $|w/\mu|>1$

$$U_1(w, \mu)=-\cos\left[\frac{1}{2}\left(w+\frac{\mu^2}{w}\right)+\frac{\pi}{2}\right]+V_1(w, \mu)$$

$$U_2(w, \mu)=-\cos\left[\frac{1}{2}\left(w+\frac{\mu^2}{w}\right)\right]+V_0(w, \mu)$$

where

$$V_n(w, \mu)=\sum_{m=0}^{\infty}(-1)^m(\mu/w)^{n+2m}J_{-n-2m}(\mu).$$

Hence we can rewrite (6) as

$$\psi=\frac{1}{2c}\frac{1}{k-\mathrm{i}\xi_1/\alpha^2}\exp\left(-\frac{\rho^2+\xi_1\xi_2}{\alpha^2}\right)[V_1(w, \mu)+\mathrm{i}V_0(w, \mu)]$$
$$-\frac{\mathrm{i}}{2c}\frac{1}{k-\mathrm{i}\xi_1/\alpha^2}\exp\left[-\frac{\rho^2+\xi_1\xi_2}{\alpha^2}+\frac{\mathrm{i}}{2}\left(w+\frac{\mu^2}{w}\right)\right]=\psi^{(1)}+\psi^{(2)}$$

where the second term $\psi^{(2)}$ describes a steady-state wave.

Making the substitution

$$\frac{\mathrm{i}}{2}\left(w+\frac{\mu^2}{w}\right)=\mathrm{i}k\left(\xi_2-\frac{\xi_1(k\rho)^2}{(k\alpha)^4+(k\xi_1)^2}\right)+\frac{\xi_1\xi_2}{\alpha^2}+\frac{(k\rho)^2\xi_1^2/\alpha^2}{(k\alpha)^4+(k\xi_1)^2}$$

we obtain

$$\psi^{(2)}=-\frac{\mathrm{i}}{2c}\frac{k\alpha^2}{(k\alpha)^2-\mathrm{i}k\xi_1}\exp\left(\mathrm{i}k\xi_2-\frac{(k\rho)^2}{(k\alpha)^2-\mathrm{i}k\xi_1}\right).$$

This is the scalar wavefunction for Brittingham's focus wave mode [2]. The focus wave mode solution $\psi^{(2)}$ exists only if $|w/\mu|>1$, hence the equation

$$\frac{\xi_2}{\alpha}\left[(k\alpha)^2+\left(\frac{\xi_1}{\alpha}\right)^2\right]=\frac{\rho^2\xi_1}{\alpha^3}$$

determines the structure of the spatial-variable domain where the focus wave mode exists. For $k\alpha^2\ll\xi_1$ the above relation becomes $\xi_1\xi_2\simeq\rho^2$, which in the τ, z representation gives the expanding sphere $\tau^2\simeq\rho^2+z^2$.

4. Discussion

Let (2) be the z-component of the electric current density pulse j_z. Then one can treat the function ψ as the Bromwich–Borgnis potential [9] (the description of electromagnetic waves in terms of this potential holds for this special type of current distribution [10]) and obtain at once the magnetic induction

$$B = B_\varphi e_\varphi = -\frac{\partial \psi}{\partial \rho} e_\varphi$$

for the TM electromagnetic wave generated by a current pulse moving with the velocity of light. The existence of such pulses was universally accepted as early as the 1960s, when the electromagnetic phenomena accompanying the absorption of x-rays were investigated (e.g. see [11] and [12]).

The description of Brittingham's focus wave mode formation can be obtained from the solution of the Goursat problem

$$4\frac{\partial^2}{\partial \xi_1 \partial \xi_2}\psi - \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\psi\right) = 0$$

$$\psi|_{\xi_1=0+} = \exp\left(ik\xi_2 - \frac{\rho^2}{\alpha^2}\right) \qquad \psi|_{\xi_2=0+} = \exp\left(-\frac{\rho^2}{\alpha^2}\right).$$

The exact solution

$$\psi = \exp\left(-\frac{\rho^2 + \xi_1\xi_2}{\alpha^2}\right)\left(J_0(\mu) + \frac{ik}{k - i\xi_1/\alpha^2}[U_1(w,\mu) + iU_2(w,\mu)]\right)$$

contains Lommel's functions of two variables w, μ (6) and the Bessel function. The second term represents a focus wave mode. Interpretation of this problem involves some difficulties if we replace the medium by homogeneous free space.

References

[1] Brittingham J N 1983 *J. Appl. Phys.* **54** 1179
[2] Sezginer A 1985 *J. Appl. Phys.* **57** 678
[3] Hillion P 1988 *J. Math. Phys.* **29** 1771
[4] Hillion P 1990 *J. Math. Phys.* **31** 3085
[5] Hillion P 1992 *J. Math. Phys.* **33** 2749
[6] Grandshteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)
[7] Smirnov V I 1981 *Course Of Higher Mathematics* vol 4, part 2 (Moscow: Nauka) p127
[8] Watson G N 1945 *Theory of Bessel Functions* (Cambridge: Cambridge University Press)
[9] Bromwich T J 1919 *Phil. Mag.* **38** 143
[10] Borisov V V and Utkin A B 1993 *J. Phys. A: Math. Gen* **26** 4081
[11] Karzas W J and Latter R 1965 *Phys. Rev.* B **137** 1369
[12] Longmire C L 1978 *IEE Trans. E.M.C.* **EMC-20** 3